Colored noise influence on the system evolution

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Abstract

We present a picture of phase transitions of the system with colored multiplicative noise. Considering the noise amplitude as the power–law dependence of the stochastic variable x^a we show the way to phase transitions disorder–order and order–disorder. The governed equations for the order parameter and one-time correlator are obtained and investigated in details. The long–time asymptotes in the disordered and ordered domains on the phase portrait of the system are defined.

Key words: Phase transitions, Stochastic processes.

I.INTRODUCTION

Considering an evolution of the stochastic system much more attention is focused on the statistical moments behaviour. Usually, a system under consideration is described in terms of order parameter η, being the first statistical moment $\langle x \rangle$, where x is a stochastic variable, and variance $\langle (\delta x)^2 \rangle$, which plays role of autocorrelator ($\delta x = x - \langle x \rangle$). Recently it has been shown that a multiplicative noise can lead a dynamical system to undergo a phase transition towards an ordered state ($\eta \neq 0$) [1, 2]. These results are obtained for extended systems within Curie–Weiss mean field theory and has been confirmed through extensive numerical simulations [3]. In these works it was

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shown that an ordered state exists only inside a window of noise intensities. Moreover, a spatial distributed system was investigated there and that kind of phase transition was stipulated by the symmetry breaking. The existence of the ordered state in the window of the noise intensity was explained by the collaboration between the multiplicative character of the noise and the presence of spatial coupling. Here we would like to describe peculiarities of the noise induced order–disorder phase transition for the zero–dimensional stochastic system.

If we want to describe the evolution of our system we need to consider temporal dependencies of the averages according to average motion equation. More simplest case of ordinary thermodynamic system in this kind approach was described in [4], where a kinetic of phase transition is presented. A system with white multiplicative noise was discussed in [5], where we describe the evolution of the system with an arbitrary noise amplitude. Because the white noise is an idealized model of fluctuations of the system parameters, here we explore in what a way the colored noise can govern the system evolution. In this letter, instead of standard approach (small noise spectral parameter extension), we will use the unified colored noise approximation developed in [6, 7, 2] to show the picture of phase transitions in stochastic systems.

II.MODEL AND BASIC EQUATIONS

In the simplest form, the problem of colored noise can be introduced considering a relevant macrovariable $x(t)$ (density of a given physical quantity) that satisfies a stochastic differential equation of the form

$$\dot{x} = f_0(x) + g_0(x)\lambda(t), \tag{1}$$

where $f_0(x)$ represents a deterministic force. A stochastic part of the evolution is defined by the amplitude $g(x)$ of fluctuations introduced through the random term $\lambda(t)$, quite often assumed to be gaussianly distributed. Without loss of generality, the deterministic part of the evolution can be chosen in polynomial Landau form

$$f_0(x) = -\frac{\partial V(x)}{\partial x}, \qquad V(x) = -\frac{\varepsilon}{2}x^2 + \frac{1}{4}x^4, \tag{2}$$

where ε is a parameter that acts as the dimensionless temperature, counted form the critical value.

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Considering a whole set of models with a typical behaviour we can use the power–law function for the noise amplitude

$$g_0(x) = x^a, \qquad a \in [0, 1]. \tag{3}$$

This kind of assumption allows us to describe systems with self–similar phase space and in particular cases we can pass to the ordinary thermodynamic system $(a = 0)$, directed percolation model $(a = 1/2)$, population dynamics and forest fires $(a = 1)$.

In the simplest case for $\lambda(t)$ we can use definition of Ornstein–Uhlenbeck process

$$\tau \dot{\lambda} = -\lambda + \xi(t), \tag{4}$$

where τ is the correlation time, $\xi(t)$ is the white noise source ($\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$).

If we take the time derivative of Eq.(1), replace first $\dot{\lambda}$ in terms of λ and ξ from Eq.(4) and then λ in terms of $\dot{x}$ and x from Eq.(1) we can obtain the non–Markovian stochastic differential equation

$$\tau \left(\ddot{x} - \frac{g_0'}{g_0}\dot{x}^2 \right) + \sigma \dot{x} = f_0 + g_0 \xi(t), \tag{5}$$

where

$$\sigma = \left[1 - \tau \left(f_0' - f_0\frac{g_0'}{g_0} \right) \right]. \tag{6}$$

According to the unified colored noise approximation we use the adiabatic elimination (neglecting $\ddot{x}$) and neglect $\dot{x}^2$.

The problem now lies in obtaining an evolution equation for the order parameter and autocorrelator. For this purpose we average reduced equation of motion. It takes the form

$$\langle \sigma(x)\dot{x} \rangle = \langle f_0(x) \rangle. \tag{7}$$

The term in right hand side can be represented as a full derivative of $dy(x) = \sigma dx$, and after averaging, following [8], we get $\langle dy(x)/dt \rangle = \langle dy(x) \rangle/dt = \langle \sigma dx \rangle/dt$. Introducing the notation for the autocorrelator $S = \langle (\delta x)^2 \rangle$ we rewrite right hand side of Eq.(7) as

$$\langle \sigma(x)\dot{x} \rangle = \dot{\eta}(\epsilon + \kappa(\eta^2 + S)) + \kappa\eta\dot{S}, \tag{8}$$

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where

$$\epsilon = 1 - \varepsilon\tau(1 - a), \qquad \kappa = \tau(3 - a). \tag{9}$$

The resulting equation for the first statistical moment reads

$$[\epsilon + \kappa(\eta^2 + S)]\dot{\eta} + \kappa\eta\dot{S} = \eta(\varepsilon - \eta^2) - 3\eta S. \tag{10}$$

Because of Eq.(10) accounts effect of correlations we need to construct an equation for the autocorrelator. For this purpose let us express following differential: $dy^2 = 2ydy + (dy)^2$. According to the aforementioned stochastic process $dy(x(t))$ we find $dy^2 = \epsilon^2 dx^2 + (2\epsilon\kappa/3)dx^4 + (\kappa/3)^2 dx^6$. Making use of the supposition $x^6 \ll 1$ we receive $\langle dy^2/dt \rangle = \epsilon^2 d\langle x^2 \rangle/dt + (2\epsilon\kappa/3)d\langle x^4 \rangle/dt$. Rewriting the Langevin equation in the form of stochastic differential equation

$$\mathrm{d}y = f_0(x)\mathrm{d}t + g_0(x)\mathrm{d}W \tag{11}$$

we immediately produce up to the first order time derivative (where for Winer process we have $(dW)^2 \sim dt$)

$$
\begin{aligned}
2\epsilon\eta &\left[\epsilon + 4\kappa\left(\frac{1}{3}\eta^2 + S\right)\right]\dot{\eta} + \epsilon[\epsilon + 4\kappa(\eta^2 + S)]\dot{S} = \\
&= 2\left[\epsilon\varepsilon(\eta^2 + S) - \left(\epsilon - \frac{\kappa\varepsilon}{3}\right)(\eta^4 + 6\eta^2 S + 3S^2)\right] + \langle x^{2a} \rangle
\end{aligned}
\tag{12}
$$

The obtained equation combines integer order averages and fractional one, namely $\langle g_0^2(x) \rangle = \langle x^{2a} \rangle$. As discussed in [5], such a kind of fractional average can be expressed in terms of order parameter and autocorrelator, according to the supposition that the probability distribution function of the initial process is a homogeneous function, i.e.

$$P(x) \approx Ax^{-2a}, \qquad A \equiv \frac{1}{2}|1 - 2a|b^{|1-2a|}, \tag{13}$$

where the cut-off parameter $b \to 0$. One yields the following definition for the fractional average [5]:

$$\langle x^{nq} \rangle = \alpha_n(q)\langle x^n \rangle^{p_n(q)} \tag{14}$$

where

$$p_n(q) = \frac{1 - 2a + nq}{1 - 2a + n},$$

$$\alpha_n(q) = A^{\frac{n(1-q)}{1-2a+n}} p_n^{-1}(q)(1 - 2a + n)^{p_n(q)-1}.$$

(15)

According to [9] a keypoint of the system with the multiplicative noise (3) is that its behaviour is governed by the magnitude of the exponent a in Eq.(3). At $1/2 < a < 1$, when the fractal dimension of the phase space $D = 2(1-a)$ is less than 1, the system is always disordered and its evolution is represented by the the autocorrelator $S(t)$. It provides $q > 1$ in Eq.(14), hence the fractional order average is represented by $S(t)$. In the case $a < 1/2$, where $D > 1$, according to the Landau theory the system can test the phase transition and in the fractional order average we account more essential contribution given by the order parameter η ($q < 1$ in Eq.(14)). Therefore, we can rewrite Eq.(14) as

$$\langle x^{2a} \rangle = \begin{cases} \alpha_1 \eta^{p_1} : & 0 < a < \frac{1}{2} \\ \alpha_2 S^{p_2} : & \frac{1}{2} < a < 1 \end{cases}$$

(16)

where

$$\alpha_1 = A^{(1-2a)p_1} p_1^{-p_1}, \quad p_1 = \frac{1}{2(1-a)},$$

$$\alpha_2 = A^{2(1-a)p_2} p_2^{-p_2}, \quad p_2 = \frac{1}{(3-2a)}.$$

(17)

III.EVOLUTION OF DISORDERED SYSTEM

Let us analyze the evolution of the disordered system, at first. Considering the case $a > 1/2$ we get $\eta(t) = 0$, hence the system evolution is defined by solutions of the following equation

$$\dot{S}\left(\frac{\epsilon}{2} + 2\kappa S\right) = S\left(\varepsilon - S\left(3 - \frac{\varepsilon\kappa}{\epsilon}\right)\right) + \alpha_2 S^{p_2}.$$

(18)

The form of time dependencies for the autocorellator is shown in Fig.1. It is seen that $S(t)$ monotonically attains the stationary magnitude determined by the equation

$$\varepsilon - \left(3 - \frac{\varepsilon\kappa}{\epsilon}\right) S_0 + \alpha_2 S_0^{p_2-1} = 0$$

(19)

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at condition $S_0 \neq -\epsilon/4\kappa$. In Fig.2 we plot steady states at different values of noise correlation times and at different values of the control parameter ε. According to Eq.(19) with ε or τ increase the stationary value S_0 rises from the minimal magnitude. Let us focus on the limit $S \ll 1$. If we put $S^{p_2} \gg S \gg S^2$ Eq.(18) gives the power–law time dependence

$$
S_{t \to 0} = Bt^{\frac{1}{1-p_2}}, \quad B = \left(\frac{2(1-p_2)\alpha_2}{\epsilon} \right)^{\frac{1}{1-p_2}} \tag{20}
$$

In the opposite case $S_0 - S \ll S_0$ one has exponential dependence $S - S_0 \propto e^{\lambda t}$, where $\lambda = (\varepsilon - S_0(3 - \varepsilon\kappa/\epsilon) + \alpha_2 p_2)/(\epsilon/2 + 2\kappa S_0)$.

IV.EVOLUTION OF ORDERING SYSTEM

We now in a position to discuss the case $a < 1/2$ of ordering system with the fractal dimension in $x - t$ space $D > 1$. It provides that the system dynamics is governed by equations for the order parameter and autocorrelator

$$
\begin{aligned}
\gamma(\eta, S)\dot{\eta} = & \\
= \quad & \eta[\epsilon - \eta^2 - 3S][\epsilon + 4\kappa(\eta^2 + S)] \\
- \quad & 2\kappa\eta \left[\varepsilon(\eta^2 + S) - \left(1 - \frac{\kappa\varepsilon}{3\epsilon}\right)(\eta^4 + 6\eta^2 S + 3S^2) \right] \\
- \quad & \kappa\epsilon\alpha_1\eta^{p_1+1},
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\beta(\eta, S)\dot{S} \quad = & \\
= \quad & \left[\varepsilon(\eta^2 + S) - \left(1 - \frac{\kappa\varepsilon}{3\epsilon}\right)(\eta^4 + 6\eta^2 S + 3S^2) \right] \\
\times \quad & [\epsilon + \kappa(\eta^2 + S)] \\
- \quad & \eta^2 \left[\epsilon + 4\kappa \left(\frac{\eta^2}{3} + S \right) \right] [\varepsilon - \eta^2 - 3S] \\
+ \quad & [\epsilon + \kappa(\eta^2 + S)]\alpha_1\eta^{p_1},
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma(\eta, S) \quad = \quad & [\epsilon + \kappa(\eta^2 + S)][\epsilon + 4\kappa(\eta^2 + S)] \\
& -2\eta^2\kappa \left[\epsilon + 4\kappa \left(\frac{\eta^2}{3} + S \right) \right],
\end{aligned} \tag{22}
$$

$$\beta(\eta, S) = \left[\frac{\epsilon}{2} + 2\kappa(\eta^2 + S)\right]\left[\epsilon + \kappa(\eta^2 + S)\right] \tag{23}$$
$$-\kappa\eta^2\left[\epsilon + 4\kappa\left(\frac{\eta^2}{3} + S\right)\right].$$

The obtained closed-loop system of differential equations can be analyzed with a help of the phase plane method. From the corresponding phase portrait shown in Fig.3a it is seen that at small values of the control parameter ε there is only one attractive point C_0 with coordinates $\eta_0 = 0$, $S_0 = (\varepsilon/3)[(1 - \varepsilon\tau(1 - a))/(1 - 2\varepsilon\tau(1 - 2a/3))]$. Increasing of the control parameter provides the appearance of saddle and attractive points with coordinates given by solutions of the stationary equations

$$
\begin{aligned}
[\epsilon \quad &- \quad \eta_0^2 - 3S_0][\epsilon + 4\kappa(\eta_0^2 + S_0)] = \tag{24}\\
&= \quad 2\kappa\left[\varepsilon(\eta_0^2 + S_0) - \left(1 - \frac{\kappa\varepsilon}{3\epsilon}\right)(\eta_0^4 + 6\eta_0^2 S + 3S_0^2)\right]\\
&+ \quad \kappa\epsilon\alpha_1\eta_0^{p_1},
\end{aligned}
$$

$$
\begin{aligned}
&\left[\varepsilon(\eta_0^2 + S_0) - \left(1 - \frac{\kappa\varepsilon}{3\epsilon}\right)(\eta_0^4 + 6\eta_0^2 S_0 + 3S_0^2)\right] \times\\
\times \quad &[\epsilon + \kappa(\eta_0^2 + S_0)] = \tag{25}\\
= \quad &\eta_0^2\left[\epsilon + 4\kappa\left(\frac{\eta_0^2}{3} + S_0\right)\right][\varepsilon - \eta_0^2 - 3S_0]\\
- \quad &[\epsilon + \kappa(\eta_0^2 + S_0)]\alpha_1\eta_0^{p_1}.
\end{aligned}
$$

The dependence of the bifurcation value of the control parameter ε_0 from the noise correlation time τ is shown in Fig.4. It illustrates the appearance of the ordered phase domain ($\eta_0 \neq 0$) at small values of the noise correlation time and small values of ε_0. So the ordered phase is realized in the window of the intensities $[\varepsilon_0^1, \varepsilon_0^2]$. Increasing of τ provides the decreasing of the domain of ordered phase. Dependencies of the steady states are shown in Fig.5,6. Here thin lines display the saddle point S and thick lines correspond to the attractive point C in Fig.3b. Some distinctive feature can be seen from Fig.5,6: the system undergoes the phase transition of the first order despite the bare x^4-potential corresponds to the continuous one. Such a kind of transition can be observed at disorder–order and order–disorder phase transitions with increasing and decreasing of the control parameter ε.

Let us discuss now time dependencies corresponding to the phase trajectories from Fig.3. Time dependencies are shown in Fig.7.

Thus, according to the phase portraits shown in Fig.5, at $\varepsilon \neq [\varepsilon_0^1, \varepsilon_0^2]$ the order parameter falls down monotonically to the point C_0, whereas the autocorrelator can vary nonmonotonically. Inside the domain $[\varepsilon_0^1, \varepsilon_0^2]$, where bifurcation occurs, we can see two domains on the phase plane. These domains correspond to the small and large values of the order parameter. At small initial values of the order parameter we receive above mentioned behaviour. At intermediate and large magnitudes the order parameter attains the attractive point C. In the vicinity of the saddle point S we get a critical slowing-down and metastable phase can exists some short time. Such a kind of behaviour can be found near the separatrix C_0SC.

Let us analyze the time dependencies of main averages analytically. Firstly, we investigate the system evolution in the disordered state (i.e. vicinity of the point C_0) for large time. According to the [5] instead of the ordinary Lapunov method we use generalized Tsallis exponent [10]:

$$e^{qt} \rightarrow \exp_q(t) \equiv [1 + (1-q)t]^{1/1-q} \tag{26}$$

with generalized Lyapunov index q. This construction is more convenient to analyze the power–law dependencies that we have in evolution equations. According to the derivation rule

$$\frac{\partial}{\partial t}\exp_q(t) = \left(\exp_q(t)\right)^q \equiv \exp_q^q(t) \tag{27}$$

and asymptotic behaviour

$$\lim_{t\to 0}\exp_q(t) \rightarrow 1 + t, \qquad \lim_{t\to\infty}\exp_q(t) \rightarrow [(1-q)t]^{1/1-q} \tag{28}$$

let us assume solutions of Eqs.(21,22) in the form

$$\eta(t) = m\exp_\mu(t), \quad S(t) = S_0 + n\exp_\nu(t). \tag{29}$$

Inserting Eq.(29) into Eq.(21) we receive up to the first order of $m, n \ll 1$:

$$(\epsilon + \kappa S_0)(\epsilon + 4\kappa S_0) = -\kappa\epsilon\alpha_1 m^{p_1}\exp_\mu^{p_1+1-\mu}(t), \tag{30}$$

where we account the singular contribution only. In the long–time limit, the function $\exp_\mu^{p_1+1-\mu}(t)$ can be taken equal to 1. It provides definition of the Lyapunov multiplier and exponent

$$
\begin{aligned}
m &= |(\epsilon + \kappa S_0)(\epsilon + 4\kappa S_0)/\kappa\epsilon\alpha_1|^{1/p_1}, \\
\mu &= 1 + \frac{1}{2(1-a)}.
\end{aligned}
\tag{31}
$$

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Considering behaviour of the autocorrelator $S(t)$ up to the first order of amplitudes m and n we obtain

$$(\epsilon/2 + 2\kappa S_0) = \exp_\nu^{1-\nu}\left(\varepsilon + \alpha_1 n^{-1} m^{p_1} \exp_\mu^{p_1}(t) \exp_\nu^{-1}(t)\right). \qquad (32)$$

In the short-time limit we assume $\exp_\nu^{1-\nu} \to 1$. The long-time asymptote yields $\exp_\mu^{p_1}(t) \exp_\nu^{-1}(t) = const \equiv p_1^{-1}$ and provides

$$\nu = 2, \qquad n = \frac{1}{\kappa \epsilon p_1} \frac{(\epsilon + \kappa S_0)(\epsilon + 4\kappa S_0)}{\epsilon/2 - \varepsilon + 2\kappa S_0}. \qquad (33)$$

Thus, according to the obtained time dependencies, we see that the order parameter behaves itself in a power–law form, i.e. $\eta(t) \propto t^{-2(1-a)}$, and the autocorrelator $S(t)$ follows to the hyperbolical dependence $S(t) \propto t^{-1}$.

If we pass through the critical value ε_0^1 then the system can be ordered and here we have to account an initial magnitude of the order parameter. Taking the initial magnitude of $\eta(0)$ larger then a critical value η_c (shown in Fig.8) we make the system to be ordered. Let us examine the system evolution to the ordered state (vicinity of the point C). Here we have to note, that we may not use the solution like generalized exponent (26). The latter is applicable at non–linearity effects which are sufficient to fix the amplitudes m and n. In the case under consideration the linear conditions are satisfied and because of power–law construction of the evolution equations we ought to use the Mellin transformations

$$\eta(t) = \eta_0 + \int m_q t^q \mathrm{d}q,$$
$$\qquad\qquad\qquad\qquad\qquad (34)$$
$$S(t) = S_0 + \int n_q t^q \mathrm{d}q.$$

Here the evolution equations can be transformed to the system of linear algebraical equations

$$A_{11} m_q + A_{12} n_q = 0,$$
$$\qquad\qquad\qquad (35)$$
$$A_{21} m_q + A_{22} n_q = 0,$$

where multipliers A_{ij} are functions of the system parameters and coordinates of the point C. The diagonal elements of the matrix $\mathbf{A}$ incorporate terms q/t, so we can rewrite $A_{ii} = A_{ii}^{(0)} + q/t$. The system (35) has solution when

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$\det |\mathbf{A}| = 0$. So if we use the following notation $c = -q/t$ we get

$$
\begin{aligned}
\eta(t) &= \eta_0 + m \exp(-ct \ln(t)), \\
S(t) &= S_0 + n \exp(-ct \ln(t)),
\end{aligned}
\tag{36}
$$

here amplitudes m, n correspond to the index $q = -ct$; c is the real number whose magnitude can be expressed from the condition $\det |\mathbf{A}| = 0$:

$$
c = \frac{1}{2}(A_{11}^{(0)} + A_{22}^{(0)}) \left(1 \pm \sqrt{1 - \frac{4(A_{11}^{(0)} A_{22}^{(0)} - A_{12} A_{21})}{(A_{11}^{(0)} + A_{22}^{(0)})^2}} \right).
\tag{37}
$$

V.SUMMARY

We have studied dynamics of noise induced order–disorder phase transition. Colored multiplicative noise contributes to realizing of two phase transitions with the control parameter increasing. We can conclude that this kind transitions are a consequence of collaboration of the nonlinearity of the system and multiplicative character of the colored noise (in the white noise limit there is only one phase transition along the axis ε [5]). We have shown that color of the noise does not change time asymptotes in the vicinity of the attractive points, corresponding to the disordered and ordered domains, comparing to the white noise limit. It changes only amplitudes of the time dependencies. We believe that these results may be found in physical and biological systems possessing a self-affine phase space.

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FIGURE CAPTIONS

Fig.1 Time dependence of the autocorrelator S at $\varepsilon = 0.6$, $\tau = 0.5$, $a = 0.8$

Fig.2 Stationary states of the system at $a = 0.8$: (a) S_0 vs. control parameter at different values of τ; (b) S_0 vs. noise correlation time at different values of ε.

Fig.3 Phase portraits at $a = 0.2$: (a) $\varepsilon = 0.1$, $\tau = 0.01$; (b) at $\varepsilon = 0.5$, $\tau = 0.2$.

Fig.4 Phase diagram.

Fig.5 Stationary states of the system at $a = 0.2$: (a) order parameter vs. control parameter ε; (b) order parameter vs. noise correlation time.

Fig.6 Stationary states of the system at $a = 0.2$: (a) autocorrelator vs. control parameter ε (b) autocorrelator vs. noise correlation time.

Fig.7 Time dependencies correspond to the different trajectories on the phase portrait in Fig.3: (a) η vs. $\ln(t)$ at $a = 0.2$, $\varepsilon = 0.5$, $\tau = 0.2$; (b) S vs. $\ln(t)$ at $a = 0.2$, $\varepsilon = 0.5$, $\tau = 0.2$.

Fig.8 Critical value of the order parameter η_c vs. noise correlation time τ at $\varepsilon = 0.5$ and different values of the noise exponent a.
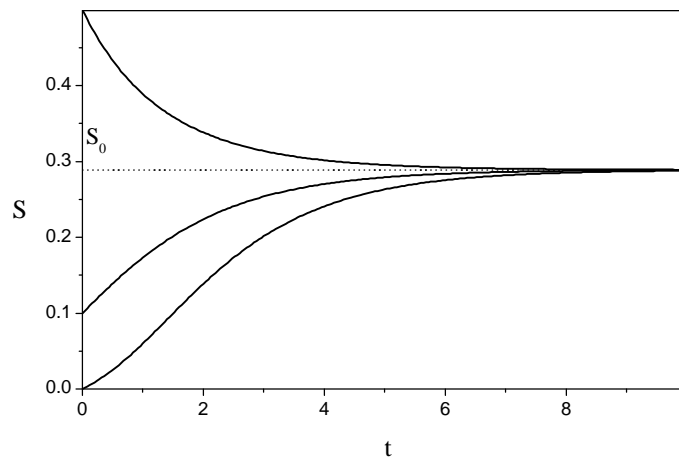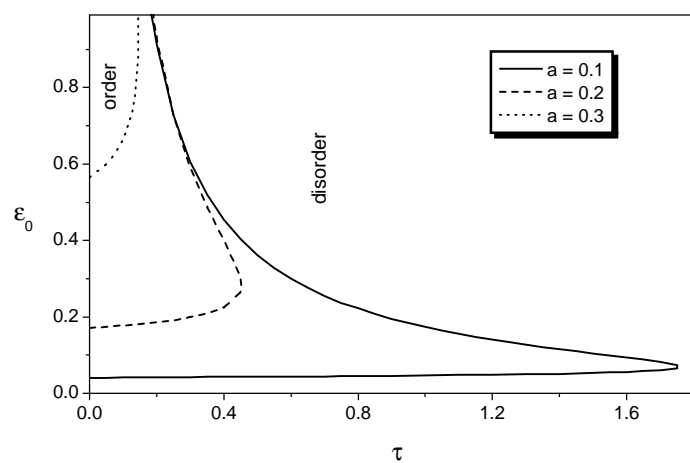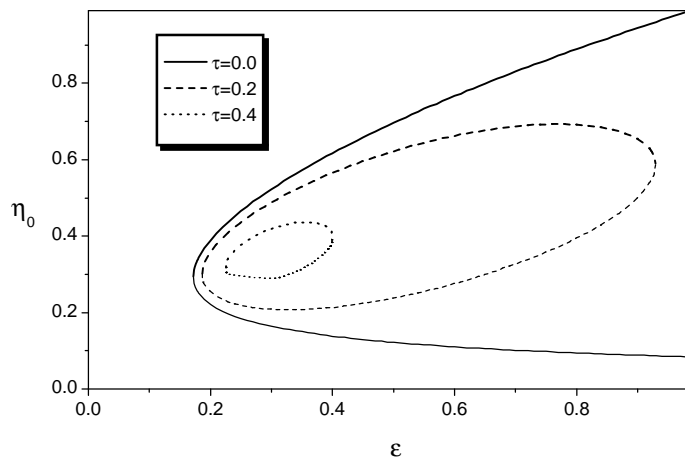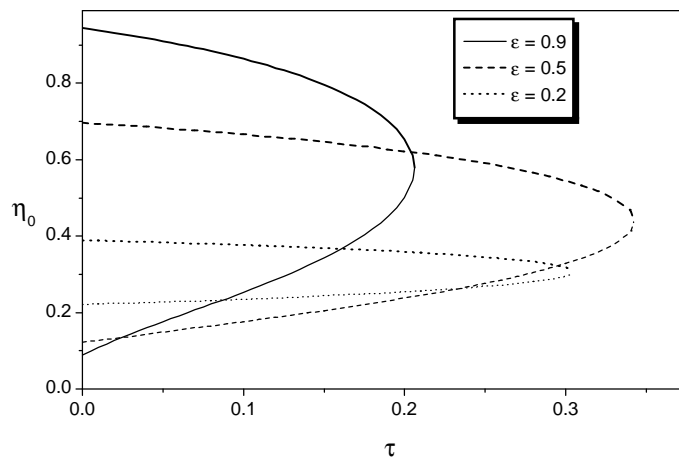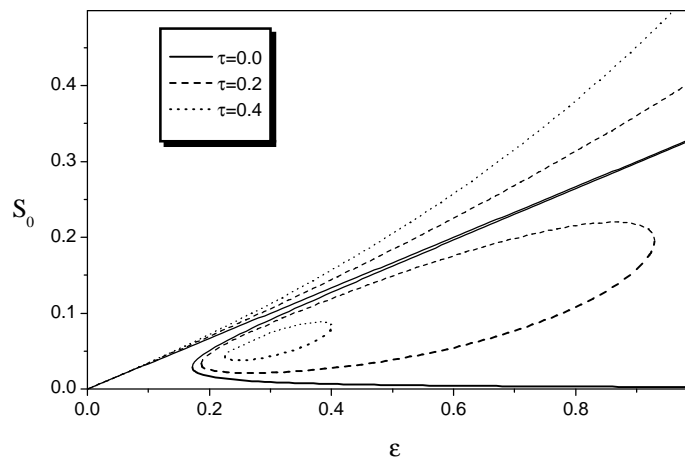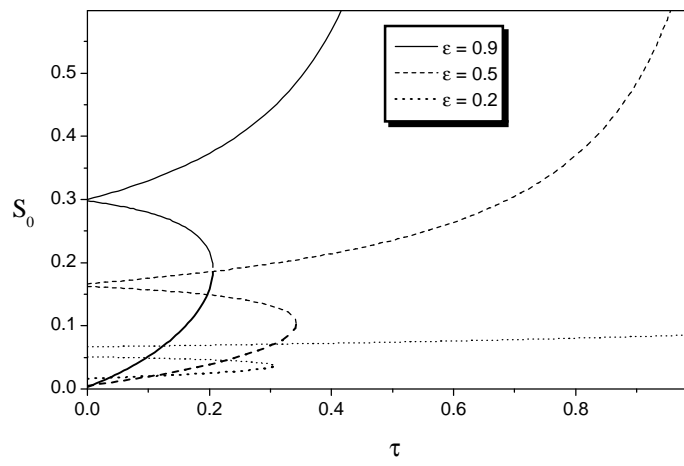
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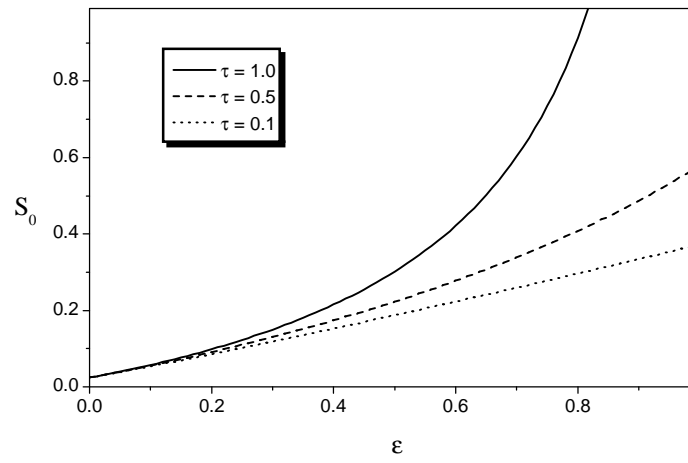
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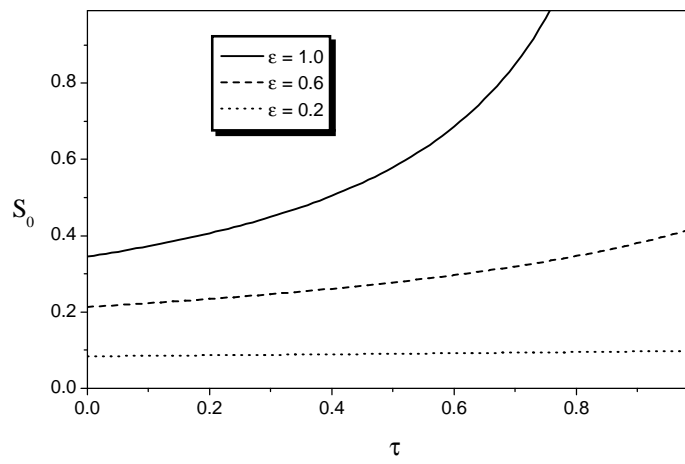
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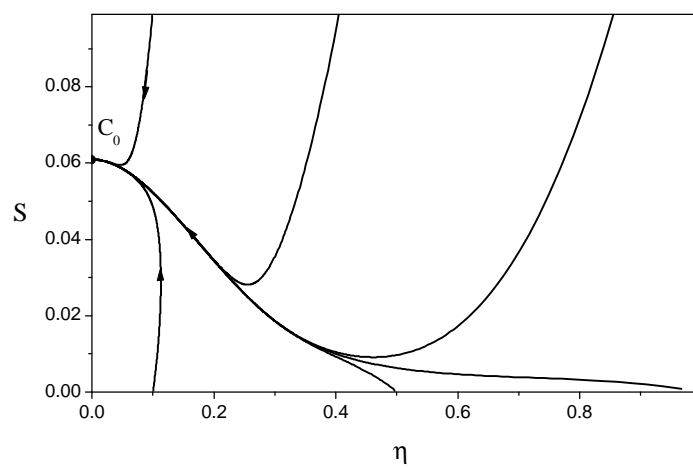
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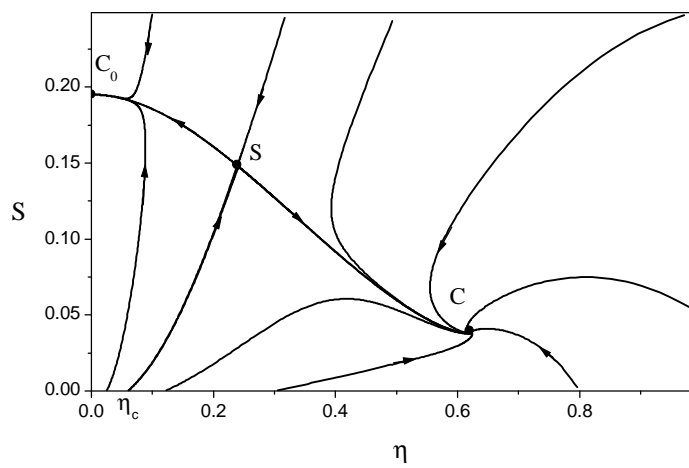
Fig.3b

Fig.4

Fig.5a

Fig.5b

Fig.6a

Fig.6b

Fig.7a

Fig.7b

Fig.8